LESSON 19 Exponents • Powers of Negative Numbers • Roots • Evaluation of Powers

19.A exponents

Often we find it necessary to indicate that a number is to be used as a factor a given number of times. For instance, if we wish to indicate that 5 is to be used as a factor seven times, we could write $5 \cdot 5 \cdot 5 \cdot 5 \cdot 5 \cdot 5 \cdot 5$. This is a cumbersome expression and mathematicians have developed a sort of mathematical shorthand called **exponential notation** that allows the expression to be written more concisely. The exponential notation for 5 used as a factor seven times is 5^7, read **"five to the seventh."** The general form of the expression is x^n, which indicates that x is to be used as a factor n times and is read **"x to the nth."**

DEFINITION OF EXPONENTIAL NOTATION

$$\underbrace{x \cdot x \cdot x \cdot \cdots \cdot x}_{n \text{ factors}} = x^n$$

In this definition, the letter x represents a real number and is called the **base** of the expression. The letter n represents a positive integer and is called the **exponent.**

For example,

$x^4 = x \cdot x \cdot x \cdot x$ The base is x and the exponent is 4.

$(-4)^3 = (-4)(-4)(-4)$ The base is (-4) and the exponent is 3.

$\left(\frac{1}{3}\right)^4 = \left(\frac{1}{3}\right)\left(\frac{1}{3}\right)\left(\frac{1}{3}\right)\left(\frac{1}{3}\right)$ The base is $\frac{1}{3}$ and the exponent is 4.

The value of an exponential expression is called a **power** of the base.

$$2^4 = (2)(2)(2)(2) = 16$$

The value of 2 used as a factor four times is 16. **We say that the fourth power of 2 is 16.** We find it convenient to sometimes use the word *power* to name the exponent. To do this, we would say that the expression

$$2^4$$

represents 2 raised to the **fourth power.**

19.B powers of negative numbers

When a positive number is raised to a positive power, the result is always a positive number.

example 19.1 Simplify: (a) 3^2 (b) 3^3 (c) 3^4 (d) -3^4

solution Each of these notations tells us that +3 is the base. The exponents 2, 3, and 4 tell us to use 3 as a factor twice, three times, and four times.

(a) $3^2 = (3)(3) = \mathbf{9}$

(b) $3^3 = (3)(3)(3) = \mathbf{27}$

(c) $3^4 = (3)(3)(3)(3) = \mathbf{81}$

(d) We must be careful here because -3^4 means the opposite of 3^4 and not $(-3)^4$.

$$-3^4 = -(3)(3)(3)(3) = \mathbf{-81}$$

When a negative number is raised to an even power, the result is always positive; and when a negative number is raised to an odd power, the result is always negative, as demonstrated in the next example.

example 19.2 Simplify: (a) $(-3)^2$ (b) $(-3)^3$ (c) $(-3)^4$ (d) $-(-3)^4$

solution The first three are straightforward.

(a) $(-3)^2 = (-3)(-3) = \mathbf{9}$

(b) $(-3)^3 = (-3)(-3)(-3) = \mathbf{-27}$

(c) $(-3)^4 = (-3)(-3)(-3)(-3) = \mathbf{81}$

(d) Be careful here. We want the opposite of $(-3)^4$.

$$-(-3)^4 = -(-3)(-3)(-3)(-3) = \mathbf{-81}$$

example 19.3 Simplify: $-3^3 - (-3)^2 + (-2)^2$

solution Be careful with the first term.

$$\begin{aligned} -3^3 - (-3)^2 + (-2)^2 &= -(3)(3)(3) - (-3)(-3) + (-2)(-2) \\ &= -27 - 9 + 4 \\ &= \mathbf{-32} \end{aligned}$$

example 19.4 Simplify: $-2^2 - 4(-3)^3 - 2(-2)^2 - 2$

solution -2^2 is -4, $(-3)^3$ is -27, and $(-2)^2$ is 4, so we get

$$\begin{aligned} -2^2 - 4(-3)^3 - 2(-2)^2 - 2 &= -4 - 4(-27) - 2(4) - 2 \\ &= -4 + 108 - 8 - 2 \\ &= \mathbf{94} \end{aligned}$$

19.C roots

If we use 3 as a factor twice, the result is 9. Thus, 3 is the positive square root of 9. We use a **radical sign** to indicate the root of a number.

$$(3)(3) = 9 \quad \text{so} \quad \sqrt{9} = 3$$

If we use 3 as a factor three times, the result is 27. Thus, 3 is the positive cube root of 27.

$$(3)(3)(3) = 27 \quad \text{so} \quad \sqrt[3]{27} = 3$$

If we use 3 as a factor four times, the result is 81. Thus, 3 is the positive fourth root of 81.

$$(3)(3)(3)(3) = 81 \quad \text{so} \quad \sqrt[4]{81} = 3$$

If we use 3 as a factor five times, the result is 243. Thus, 3 is the positive fifth root of 243.

$$(3)(3)(3)(3)(3) = 243 \quad \text{so} \quad \sqrt[5]{243} = 3$$

Because $(-3)(-3)$ equals $+9$, we say that -3 is the negative square root of 9. Because $(-3)(-3)(-3)(-3)$ equals $+81$, we say that -3 is the negative fourth root of 81. **If n is an even number, every positive real number has a positive nth root and a negative nth root. We use the radical sign to designate the positive even root.** To designate the negative even root, we also use a minus sign.

$$\sqrt{9} = 3 \qquad -\sqrt{9} = -3 \qquad \sqrt[4]{81} = 3 \qquad -\sqrt[4]{81} = -3$$

The number under the radical sign is called the **radicand,** and the little number that designates the root is called the **index.** If n is an odd number, every positive number has exactly one positive real nth root. Also, every negative number has exactly one negative real nth root.

$$\sqrt[3]{27} = 3 \qquad \sqrt[3]{-27} = -3$$

In this book, we will consider even roots of positive numbers and odd roots of both positive and negative numbers. Even roots of negative numbers will be discussed in the next book in this series.

example 19.5 Simplify: (a) $\sqrt{64}$ (b) $\sqrt[4]{16}$ (c) $\sqrt[3]{-27}$ (d) $-\sqrt{81}$

solution (a) The notation $\sqrt{64}$ designates the positive number which used as a factor twice has a product of 64. The answer is **8** because 8 times 8 equals 64. The notations $\sqrt[2]{64}$ and $\sqrt{64}$ both designate the positive square root of 64. If the index is not written, it is understood to be 2.

$$\sqrt{64} = \mathbf{8}$$

(b) The fourth root of 16 is 2 because

$$\sqrt[4]{16} = \mathbf{2} \qquad \text{because} \qquad (2)(2)(2)(2) = 16$$

(c) Every real number has exactly one real nth root, where n is odd.

$$\sqrt[3]{-27} = \mathbf{-3} \qquad \text{because} \qquad (-3)(-3)(-3) = -27$$

(d) The square root of 81 is 9. We want the opposite of this.

$$-\sqrt{81} = \mathbf{-9}$$

19.D evaluation of powers

Evaluation of expressions with exponents is straightforward when the replacements of the variables are all positive numbers. To evaluate

$$yx^2m^3$$

with $y = 3$, $x = 4$, and $m = 2$, we proceed as follows:

$$(3)(4)^2(2)^3 = (3)(16)(8) = \mathbf{384}$$

We must be careful, however, when the expressions contain minus signs or when some replacement values of the variables are negative numbers.

example 19.6 If $a = -2$, what is the value of each of the following?

(a) a^2 (b) $-a^2$ (c) $-a^3$ (d) $(-a)^3$

solution (a) a^2 means a times a, or $(-2)(-2) = \mathbf{+4}$.

(b) $-a^2$ asks for the opposite of a^2, or $-(-2)(-2) = \mathbf{-4}$.

(c) $-a^3$ means the opposite of a times a times a, or $-(-2)(-2)(-2) = \mathbf{+8}$.

(d) $(-a)^3$ means $(-a)(-a)(-a) = (2)(2)(2) = \mathbf{+8}$.

example 19.7 Evaluate: x^2z^3y if $x = 2$, $z = 3$, and $y = -2$

solution We replace x with 2, z with 3, and y with -2.

$$(2)^2(3)^3(-2) = (4)(27)(-2) = \mathbf{-216}$$

example 19.8 Evaluate: $pm^2 - z^3$ if $p = 1$, $m = -4$, and $z = -2$

solution We replace p with 1, m with -4, and z with -2.

$$(1)(-4)^2 - (-2)^3 = (1)(16) - (-8) = 16 + 8 = \mathbf{24}$$

practice Simplify:

a. $(-2)^2$ **b.** -2^2 **c.** $-3^3 - (-2)^2 - 2^2$

d. $\sqrt[3]{-64}$ **e.** $(-3)^3 + \sqrt[3]{-27}$ **f.** $-3^2 - \sqrt[3]{-8} - \sqrt{16}$

Evaluate:

g. x^2z^3y if $x = -3$, $z = -2$, and $y = -2$

h. $b^2 - 4ac$ if $b = -4$, $a = -3$, and $c = -5$

problem set 19

1. *(18)* What kinds of terms may be added in an algebraic expression?

2. *(13)* Is the product of 7 positive numbers and 6 negative numbers a positive number or a negative number?

3. *(2)* (a) What is a right triangle?

(b) What is an acute triangle?

(c) What is an obtuse triangle?

(d) What is an equiangular triangle?

4. *(4)* Use two unit multipliers to convert 10,000 inches to miles.

5. *(10)* Use two unit multipliers to convert 15,000 square feet to square miles.

6. *(8)* The area of a rectangle is 92 square centimeters. The length of the rectangle is 23 centimeters. Find the width of the rectangle.

7. *(8)* The length of the base of a scalene triangle is 9 meters. The height of the scalene triangle is 14 meters. Find the area of the scalene triangle.

Simplify:

8. *(19)* $(-4)^2$ **9.** *(19)* -4^2 **10.** *(19)* $-2^2 + (-2)^2$

11. *(19)* $-3^2 - (-3)^2$ **12.** *(19)* $\sqrt[3]{8}$ **13.** *(19)* $\sqrt[3]{-8}$

Evaluate:

14. *(19)* x^2y^3z if $x = 3$, $y = -2$, and $z = 4$

15. *(19)* $-x^2 - y^3$ if $x = -3$ and $y = -2$

Simplify by adding like terms:

16. *(18)* $xym - 3ymx - 4xmy - 3my + 2ym$ **17.** *(18)* $a - 3 - 7a + 2a - 6ax + 4xa - 5$

Expand by using the distributive property:

18. *(17)* $x(4 - ap)$ **19.** *(17)* $(5p - 2c)4xy$ **20.** *(17)* $4k(2c - a + 3m)$

Evaluate:

21. *(14)* $-x(a - 3x) + x$ if $a = 3$ and $x = 4$

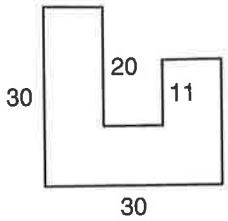
22. *(14)* $-(a - x)(x - a)$ if $a = -5$ and $x = 3$

23. *(16)* $-a[(x - a) + (2x + a)]$ if $a = -4$ and $x = 3$

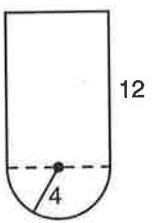
Simplify:

24. *(12)* $-3(4 - 3) - 3 - |-3|$ **25.** *(10)* $\dfrac{8 + 2 - 3(2)}{3(2) - 6}$

26. (3) Find the perimeter of this figure. All angles are right angles. Dimensions are in kilometers.



27. (8) Find the area of this figure. Corners that look square are square. Dimensions are in inches.



Simplify:

28. (1) $9\frac{2}{15} - 3\frac{1}{5} - 3\frac{1}{3}$

29. (4) $12.16608 \div 3.04$

30. (15) Find the surface area of this right triangular prism. Dimensions are in feet.

